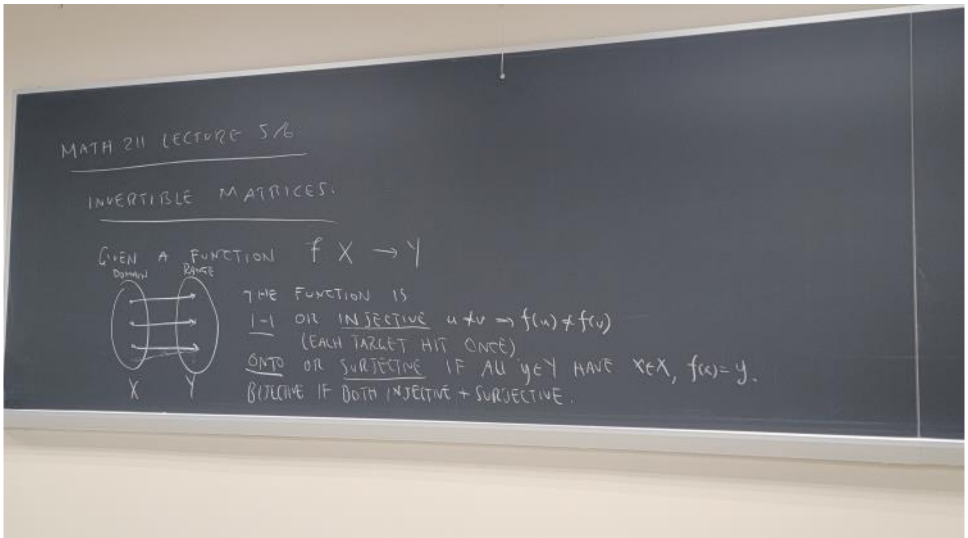
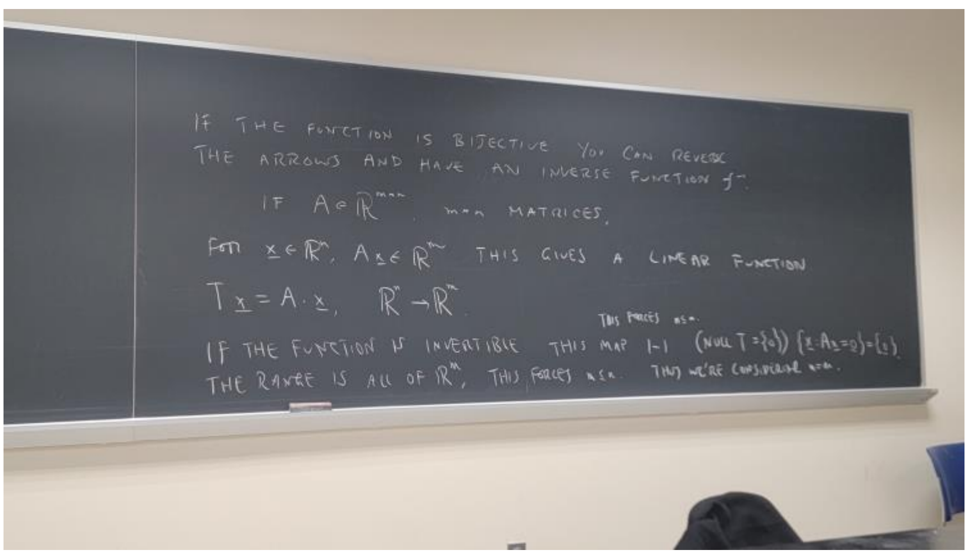
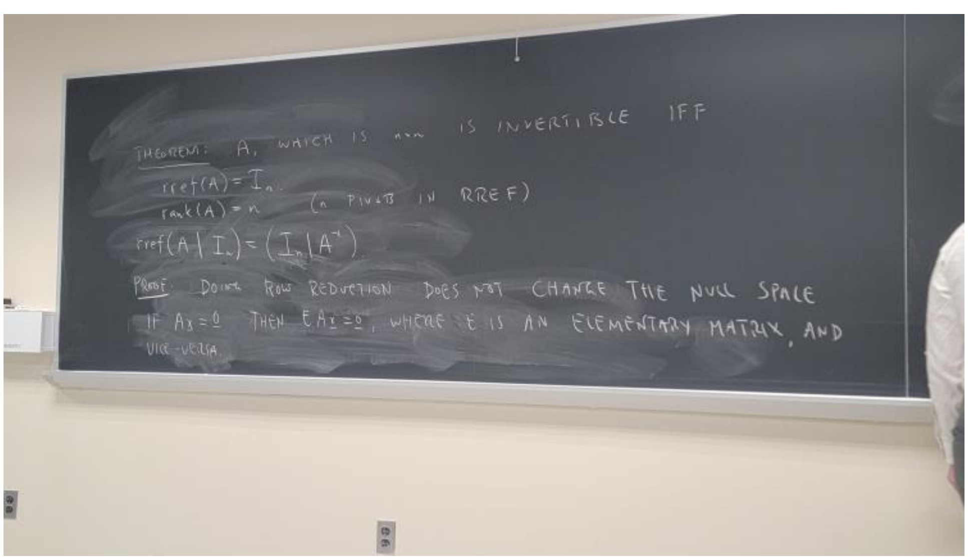
# Lecture #5

Thursday, February 9, 2023 1:45 PM



THIS MEANS

$$\mathrm{null}(A) = \{ \underline{x} : A\underline{x} = \underline{0} \}$$
$$= \{ \underline{x} : \mathrm{rref}(A)\, \underline{x} = \underline{0} \}$$
$$= \mathrm{null}\ \mathrm{rref}(A).$$

TO BE INVERTIBLE, $\mathrm{null}\ A = \{\underline{0}\}$ (SINGLE PREIMAGE OF $\underline{0}$)

SO: $\mathrm{rref}(A)$ HAS $n$ PIVOTS, IS $I_n$, HAS rank $n$.

THIS SHOWS $A$ INVERTIBLE $\rightarrow$ $\mathrm{rref}(A) = I_n$.

FOR THE REVERSE DIRECTION, IF $\mathrm{rref}(A) = I_n$,

THAT MEANS THAT THERE ARE ELEMENTARY MATRICES

$E_1, \ldots, E_k$, $\quad E_k E_{k-1} \cdots E_1 \cdot A = I_n$.

THUS $A^{-1} = E_k E_{k-1} \cdots E_1$, $A$ INVERTIBLE

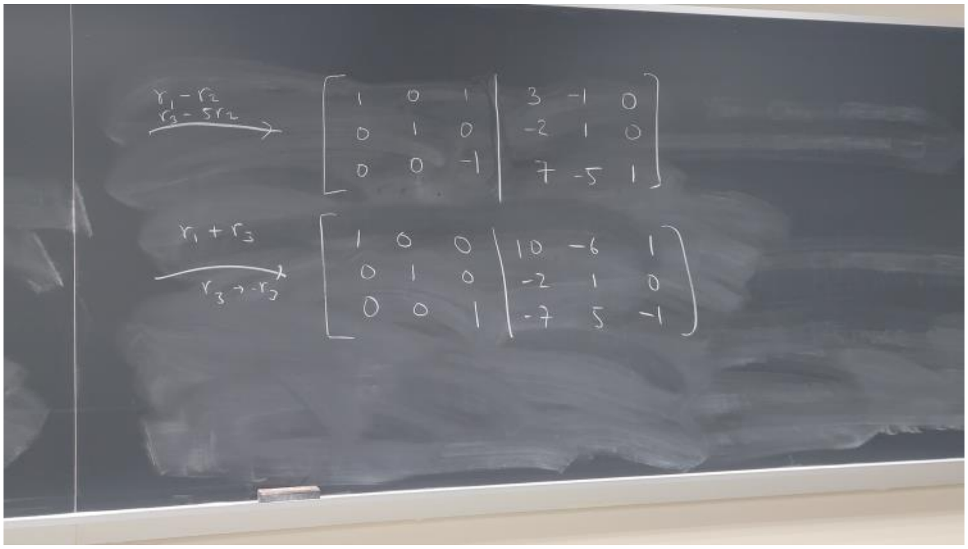
$$A = E_1^{-1} E_2^{-1} \cdots E_k^{-1}.$$

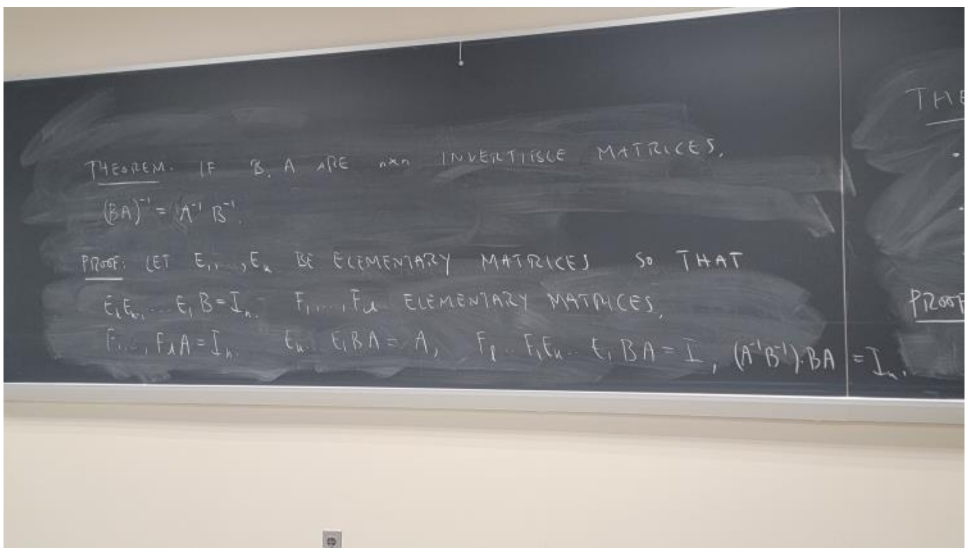
ALSO $AA^{-1} = A^{-1}A = I_n$.

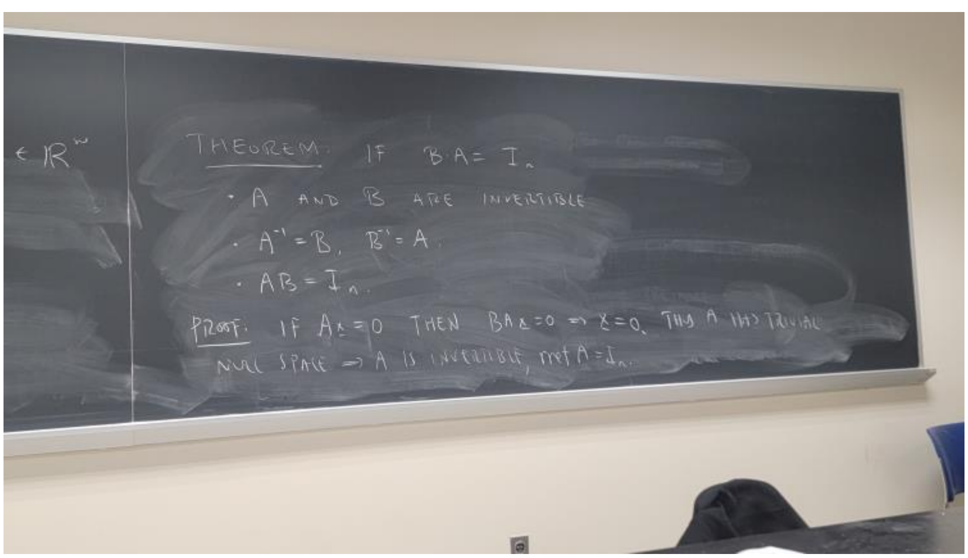
$$\mathrm{rref}(A \mid I_n)$$

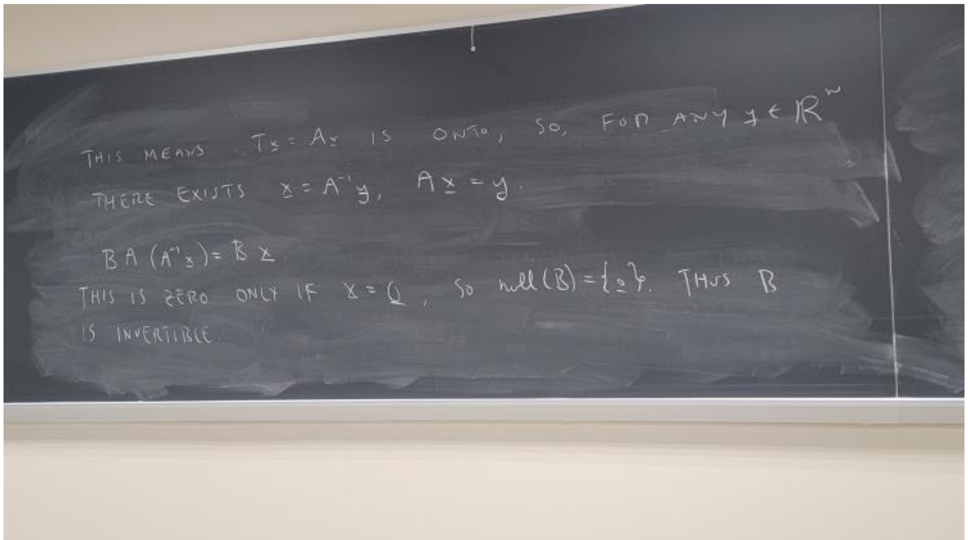
SAY $E_k E_{k-1} \cdots E_1 A = I_n$

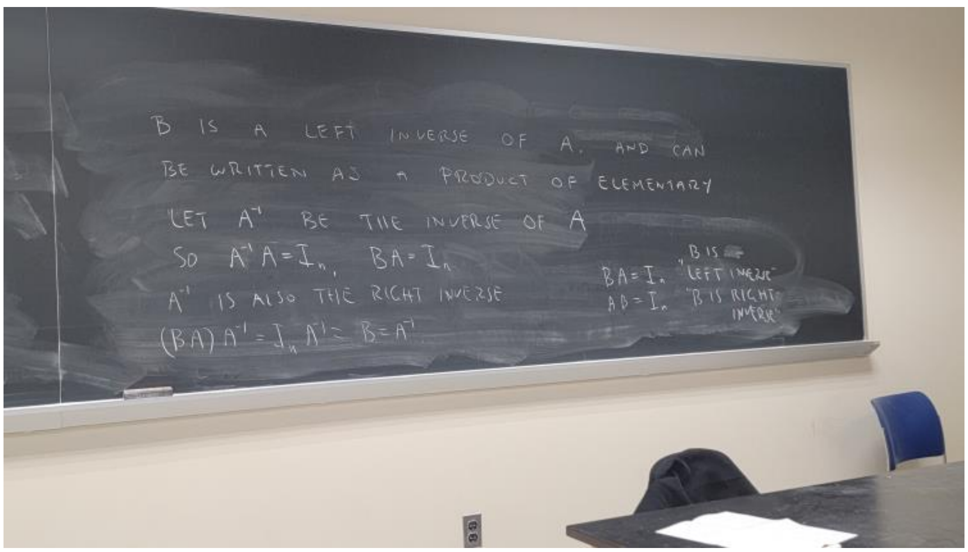
$$E_k E_{k-1} \cdots E_1 [A \mid I_n]$$
$$= [E_k E_{k-1} \cdots E_1 A \mid E_k E_{k-1} \cdots E_1 I_n]$$
$$= [I_n \mid A^{-1}]$$

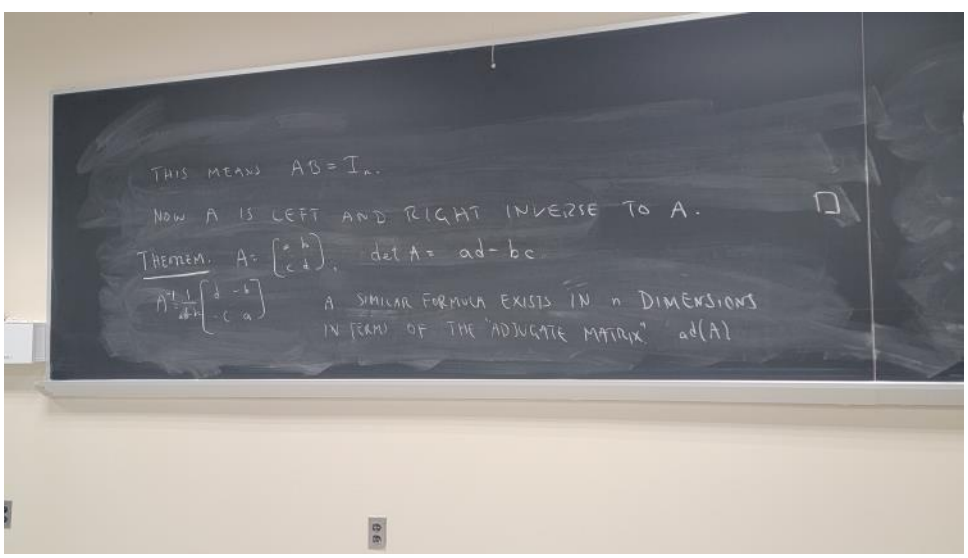

$r_1 - r_2$
$r_3 - 5r_2$
$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 3 & -1 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & -1 & 7 & -5 & 1 \end{array}\right]$
$r_1 + r_3$
$r_3 \to -r_3$
$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 10 & -6 & 1 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & -7 & 5 & -1 \end{array}\right]$

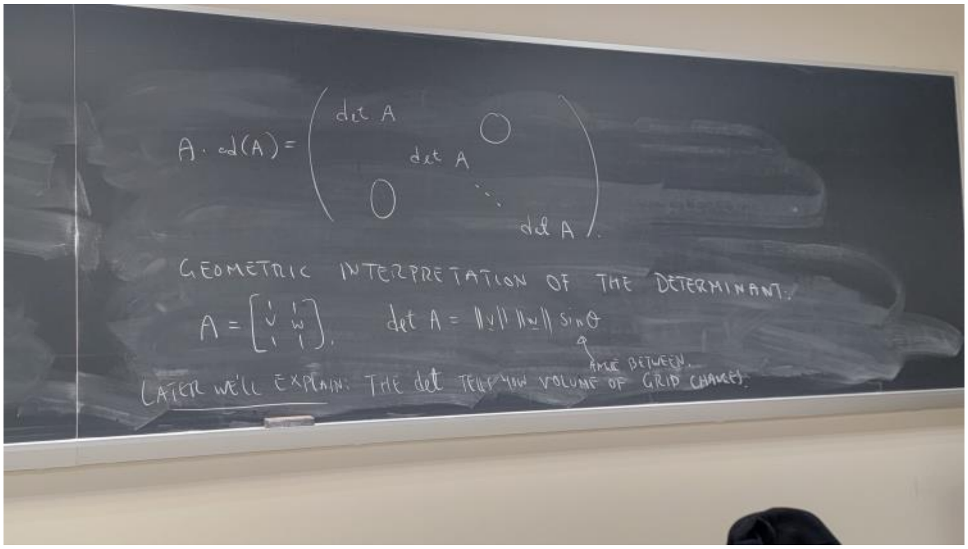

THEOREM. IF B, A ARE $n \times n$ INVERTIBLE MATRICES,
$(BA)^{-1} = A^{-1}B^{-1}$.
PROOF: LET $E_1, \dots, E_k$ BE ELEMENTARY MATRICES SO THAT
$E_k E_{k-1} \cdots E_1 B = I_n$. $F_1, \dots, F_l$ ELEMENTARY MATRICES,
$F_1 \cdots F_l A = I_n$. $E_k \cdots E_1 BA = A$, $F_l \cdots F_1 E_k \cdots E_1 BA = I$, $(A^{-1}B^{-1}) \cdot BA = I_n$.

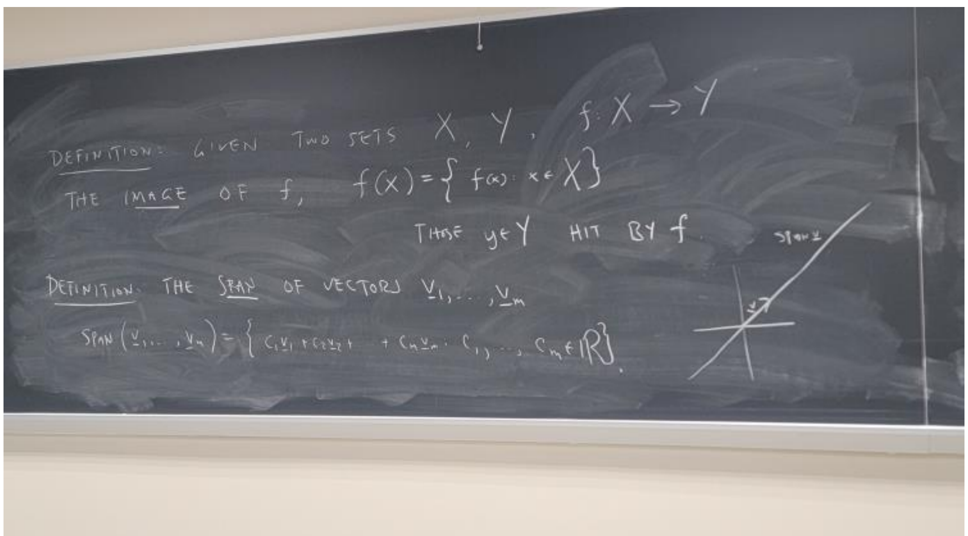

$\in \mathbb{R}^n$
THEOREM. IF $B \cdot A = I_n$
· A AND B ARE INVERTIBLE
· $A^{-1} = B$, $B^{-1} = A$.
· $AB = I_n$.
PROOF. IF $A\underline{x} = 0$ THEN $BA\underline{x} = 0 \Rightarrow \underline{x} = 0$. THUS A HAS TRIVIAL
NULL SPACE $\Rightarrow$ A IS INVERTIBLE, rref $A = I_n$.

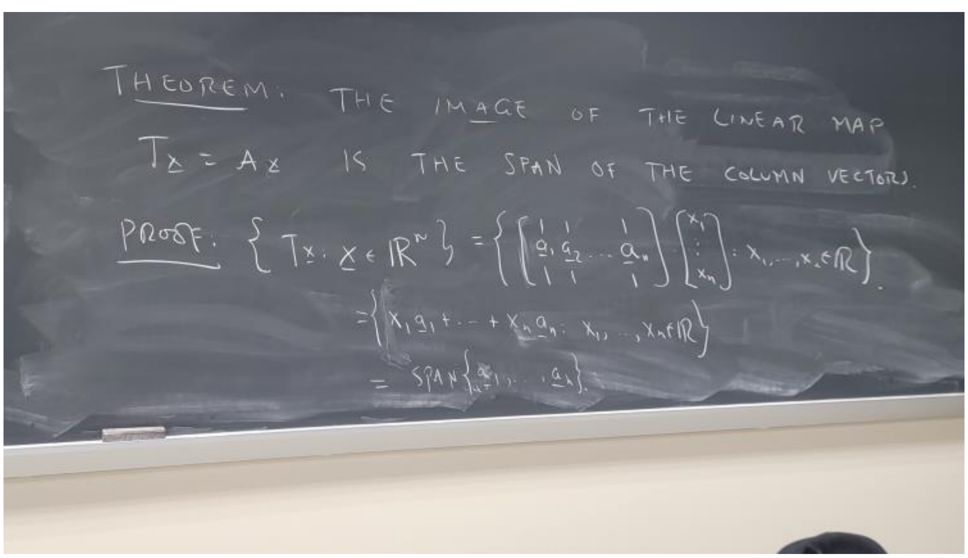

THIS MEANS $T\underline{x} = A\underline{x}$ IS ONTO, SO, FOR ANY $\underline{y} \in \mathbb{R}^n$
THERE EXISTS $\underline{x} = A^{-1}\underline{y}$, $A\underline{x} = \underline{y}$.
$BA(A^{-1}\underline{x}) = B\underline{x}$
THIS IS ZERO ONLY IF $\underline{x} = \underline{0}$, SO $\mathrm{null}(B) = \{\underline{0}\}$. THUS $B$
IS INVERTIBLE.

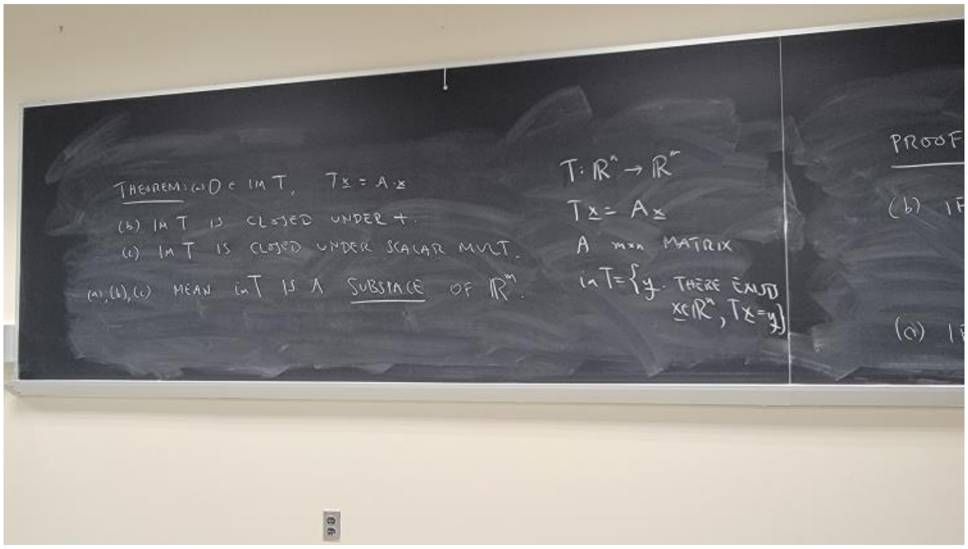

B IS A LEFT INVERSE OF A. AND CAN
BE WRITTEN AS A PRODUCT OF ELEMENTARY
LET $A^{-1}$ BE THE INVERSE OF A
SO $A^{-1}A = I_n$, $BA = I_n$
$A^{-1}$ IS ALSO THE RIGHT INVERSE
$(BA)A^{-1} = I_n A^{-1} = B = A^{-1}$
$BA = I_n$ "B IS LEFT INVERSE"
$AB = I_n$ "B IS RIGHT INVERSE"

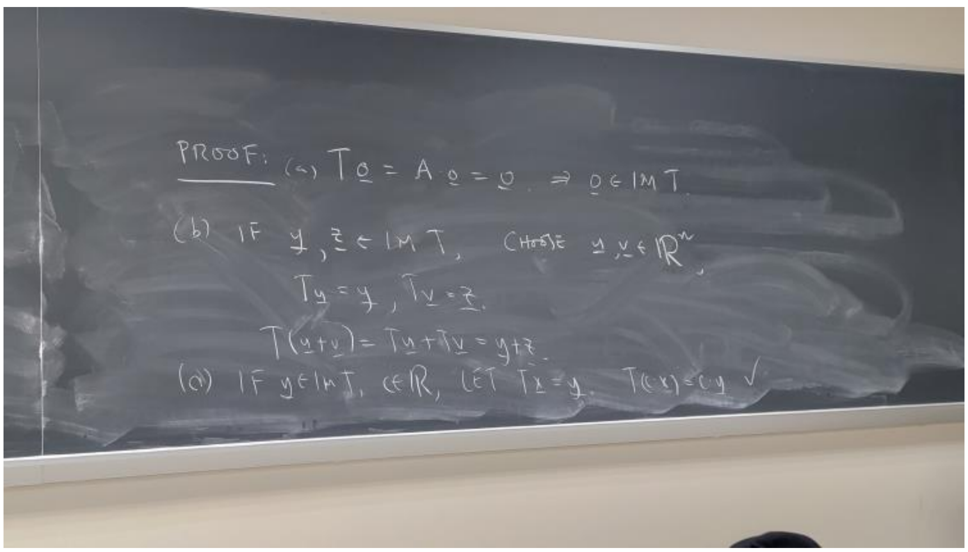

THIS MEANS $AB = I_n$.
NOW A IS LEFT AND RIGHT INVERSE TO A.
THEOREM. $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $\det A = ad - bc$
$A^{-1} = \frac{1}{\det A}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$
A SIMILAR FORMULA EXISTS IN n DIMENSIONS
IN TERMS OF THE "ADJUGATE MATRIX" ad(A)

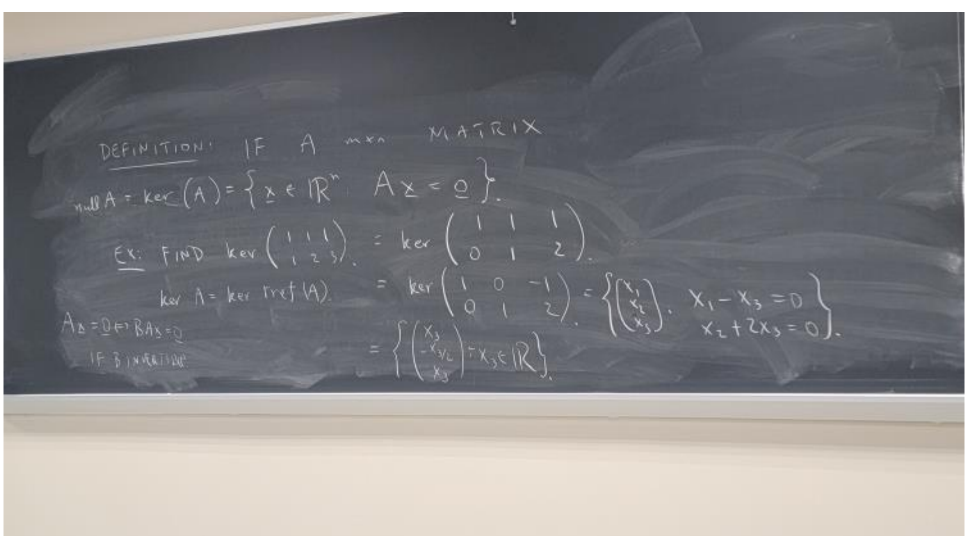

$A \cdot \mathrm{ad}(A) = \begin{pmatrix} \det A & & & 0 \\ & \det A & & \\ & & \ddots & \\ 0 & & & \det A \end{pmatrix}.$
GEOMETRIC INTERPRETATION OF THE DETERMINANT:
$A = \begin{bmatrix} | & | \\ \underline{v} & \underline{w} \\ | & | \end{bmatrix}.$ $\det A = \|\underline{v}\| \|\underline{w}\| \sin\theta$
ANGLE BETWEEN.
LATER WE'LL EXPLAIN: THE det TELLS YOU VOLUME OF GRID CHANGES.

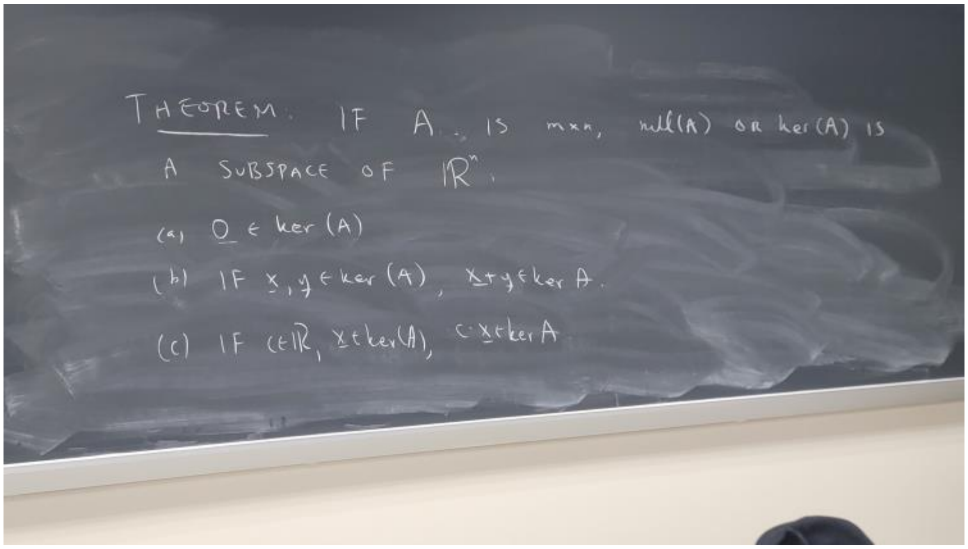

DEFINITION: GIVEN TWO SETS $X, Y$, $f: X \to Y$
THE IMAGE OF $f$, $f(X) = \{ f(x) : x \in X \}$
THOSE $y \in Y$ HIT BY $f$.
DEFINITION: THE SPAN OF VECTORS $\underline{v}_1, \ldots, \underline{v}_m$
$\mathrm{SPAN}(\underline{v}_1, \ldots, \underline{v}_m) = \{ c_1\underline{v}_1 + c_2\underline{v}_2 + \cdots + c_m\underline{v}_m : c_1, \ldots, c_m \in \mathbb{R} \}.$
SPAN $\underline{v}$
$\underline{v}$

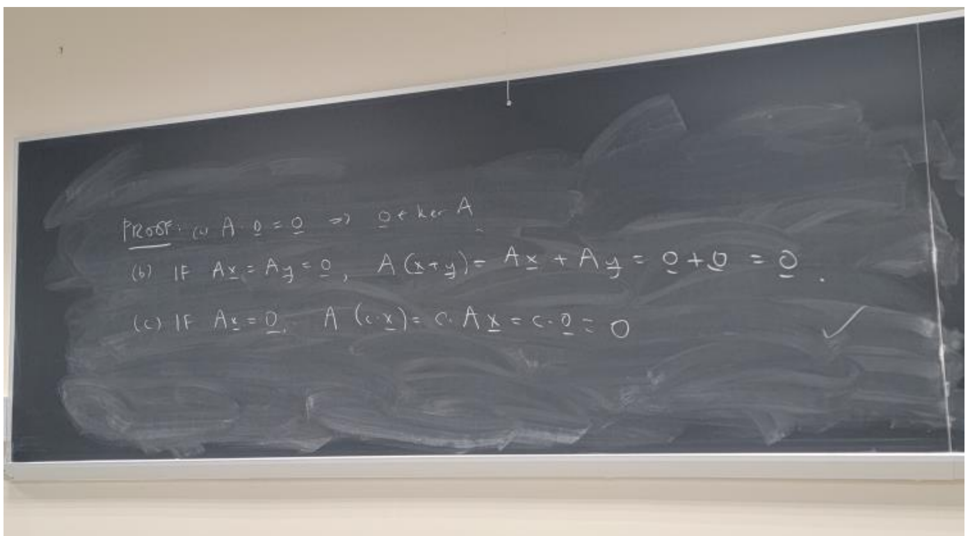

THEOREM: THE IMAGE OF THE LINEAR MAP
$T\underline{x} = A\underline{x}$ IS THE SPAN OF THE COLUMN VECTORS.
PROOF: $\{ T\underline{x} : \underline{x} \in \mathbb{R}^n \} = \left\{ \begin{bmatrix} | & | & & | \\ \underline{a}_1 & \underline{a}_2 & \cdots & \underline{a}_n \\ | & | & & | \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} : x_1, \ldots, x_n \in \mathbb{R} \right\}.$
$= \{ x_1\underline{a}_1 + \cdots + x_n\underline{a}_n : x_1, \ldots, x_n \in \mathbb{R} \}$
$= \mathrm{SPAN}\{ \underline{a}_1, \ldots, \underline{a}_n \}.$

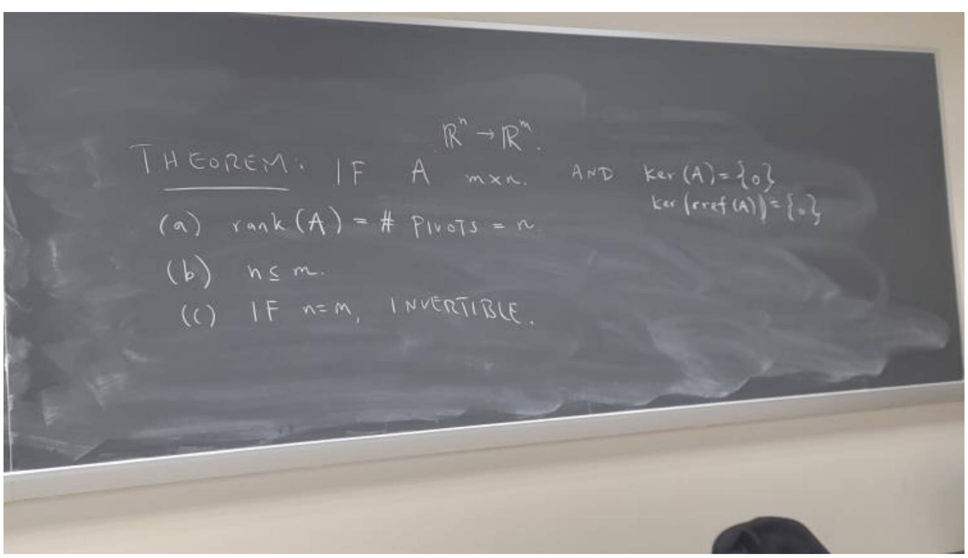

THEOREM: (a) $0 \in \operatorname{IM} T$, $T\underline{x} = A \cdot \underline{x}$
(b) IM T IS CLOSED UNDER +.
(c) IM T IS CLOSED UNDER SCALAR MULT.
(a), (b), (c) MEAN im T IS A SUBSPACE OF $\mathbb{R}^m$.
$T: \mathbb{R}^n \to \mathbb{R}^m$
$T\underline{x} = A\underline{x}$
A m×n MATRIX
$\operatorname{im} T = \{\underline{y}$. THERE EXIST $\underline{x} \in \mathbb{R}^n, T\underline{x} = \underline{y}\}$
PROOF
(b) IF
(c) IF

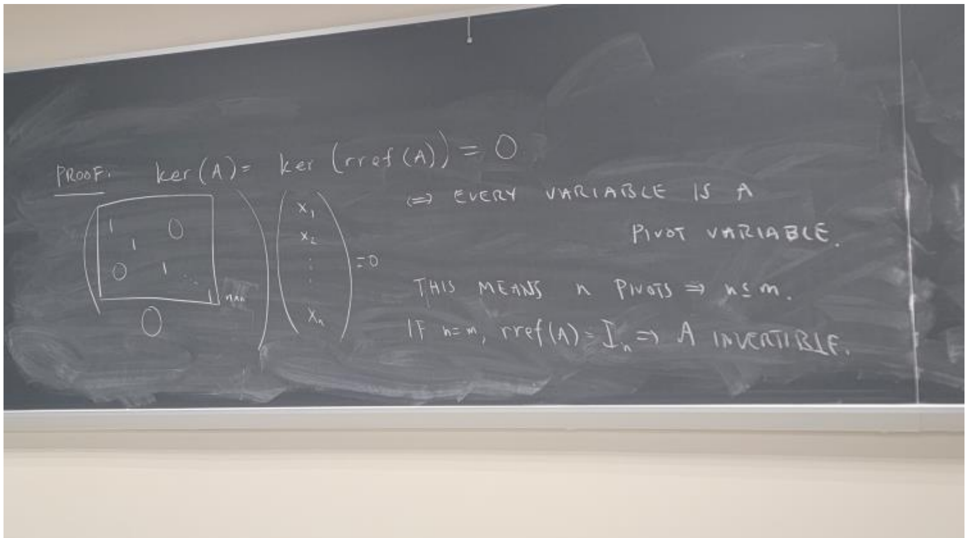

PROOF: (a) $T\underline{0} = A \cdot \underline{0} = \underline{0}$. $\Rightarrow \underline{0} \in \operatorname{IM} T$.
(b) IF $\underline{y}, \underline{z} \in \operatorname{IM} T$, CHOOSE $\underline{u}, \underline{v} \in \mathbb{R}^n$,
$T\underline{u} = \underline{y}$, $T\underline{v} = \underline{z}$.
$T(\underline{u}+\underline{v}) = T\underline{u} + T\underline{v} = \underline{y}+\underline{z}$.
(c) IF $\underline{y} \in \operatorname{Im} T$, $c \in \mathbb{R}$, LET $T\underline{x} = \underline{y}$. $T(c\underline{x}) = c\underline{y}$ ✓

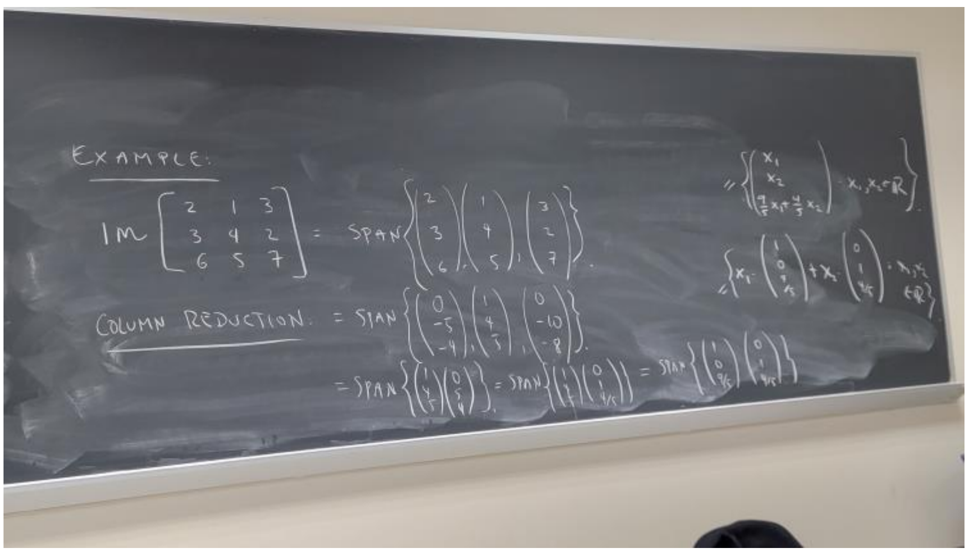

DEFINITION: IF A m×n MATRIX
$\operatorname{null} A = \ker(A) = \{\underline{x} \in \mathbb{R}^n \ A\underline{x} = \underline{0}\}$.
EX: FIND $\ker \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{pmatrix} = \ker \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix}$.
$\ker A = \ker \operatorname{rref}(A)$. $= \ker \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \end{pmatrix} = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} . \begin{matrix} x_1 - x_3 = 0 \\ x_2 + 2x_3 = 0 \end{matrix} \right\}$.
$A\underline{x} = \underline{0} \Leftrightarrow BA\underline{x} = \underline{0}$
IF B INVERTIBLE
$= \left\{ \begin{pmatrix} x_3 \\ -x_3/2 \\ x_3 \end{pmatrix} : x_3 \in \mathbb{R} \right\}$.

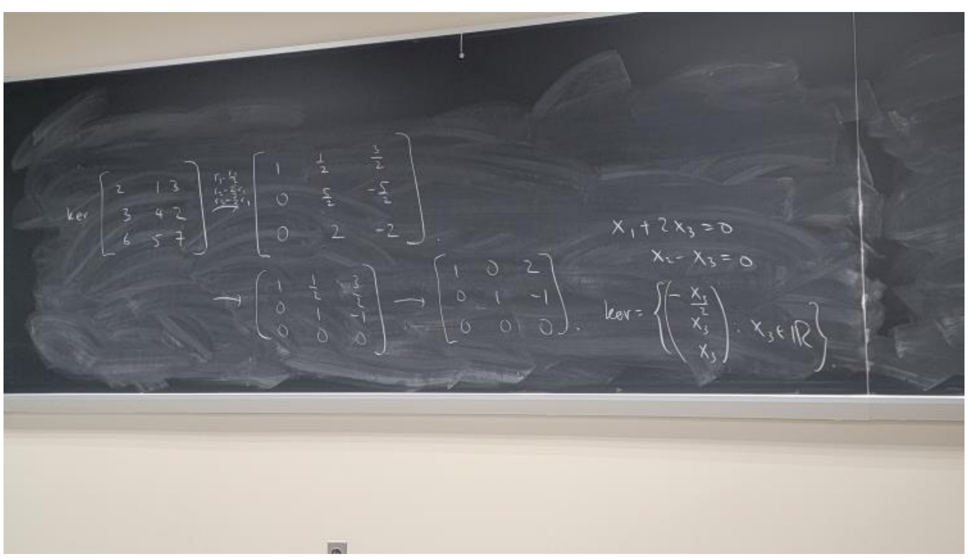

THEOREM. IF A IS $m \times n$, null(A) OR ker(A) IS
A SUBSPACE OF $\mathbb{R}^n$.
(a) $\underline{0} \in \ker(A)$
(b) IF $\underline{x}, \underline{y} \in \ker(A)$, $\underline{x}+\underline{y} \in \ker A$.
(c) IF $c \in \mathbb{R}$, $\underline{x} \in \ker(A)$, $c \cdot \underline{x} \in \ker A$

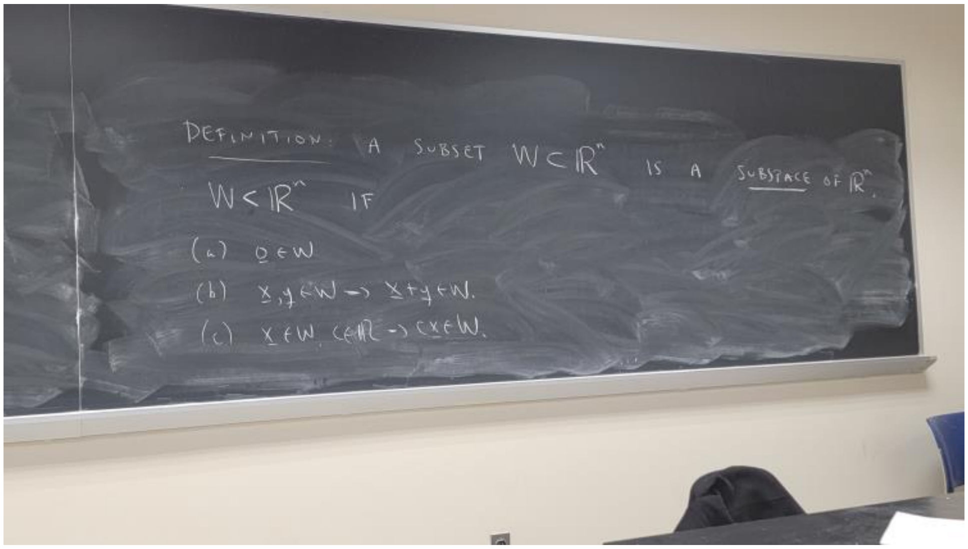

PROOF: (a) $A \cdot \underline{0} = \underline{0} \Rightarrow \underline{0} \in \ker A$
(b) IF $A\underline{x} = A\underline{y} = \underline{0}$, $A(\underline{x}+\underline{y}) = A\underline{x} + A\underline{y} = \underline{0} + \underline{0} = \underline{0}$.
(c) IF $A\underline{x} = \underline{0}$, $A(c \cdot \underline{x}) = c \cdot A\underline{x} = c \cdot \underline{0} = 0$

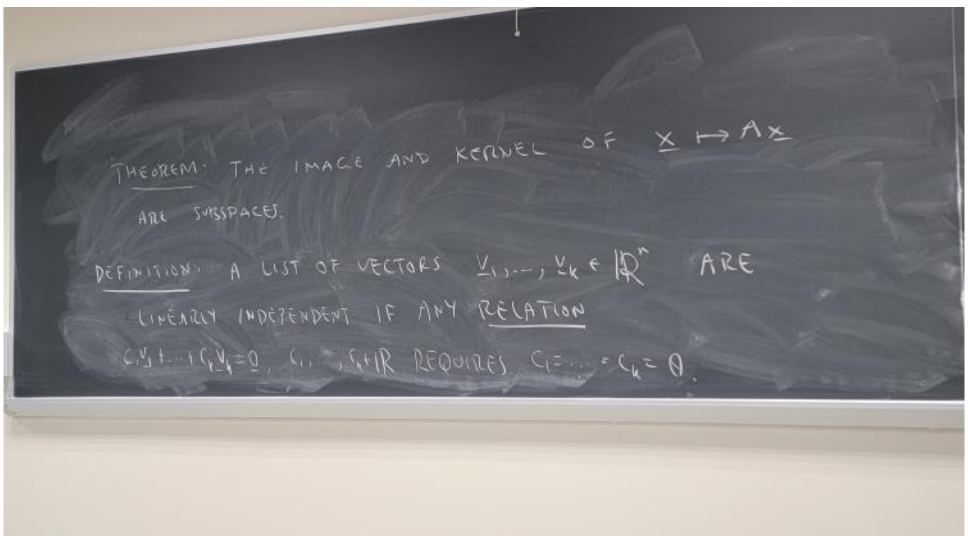

$\mathbb{R}^n \to \mathbb{R}^m$.
THEOREM: IF A $m \times n$. AND $\ker(A) = \{0\}$
$\ker(\mathrm{rref}(A)) = \{0\}$
(a) $\mathrm{rank}(A) = \#$ PIVOTS $= n$.
(b) $n \leq m$.
(c) IF $n = m$, INVERTIBLE.

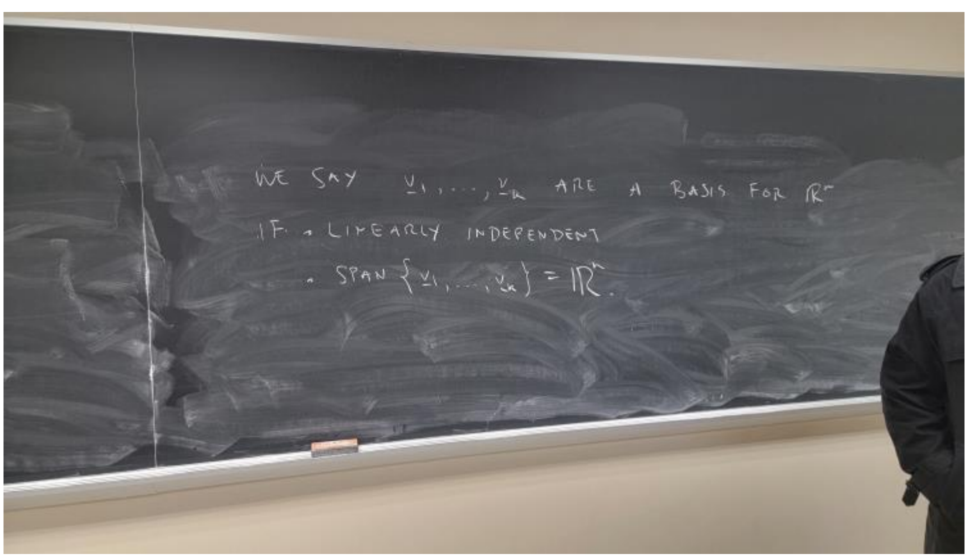

PROOF: ker(A) = ker(rref(A)) = 0
⟺ EVERY VARIABLE IS A PIVOT VARIABLE.
THIS MEANS n PIVOTS ⇒ n ≤ m.
IF n = m, rref(A) = I_n ⇒ A INVERTIBLE.


EXAMPLE:
Im [2 1 3; 3 4 2; 6 5 7] = SPAN{(2, 3, 6), (1, 4, 5), (3, 2, 7)}.
COLUMN REDUCTION: = SPAN{(0, -5, -4), (1, 4, 5), (0, -10, -8)}.
= SPAN{(1, 0, 9/5), (0, 1, 4/5)}


ker [2 1 3; 3 4 2; 6 5 7]
[1 1/2 3/2; 0 5/2 -5/2; 0 2 -2]
→ [1 0 2; 0 1 -1; 0 0 0].
x₁ + 2x₃ = 0
x₂ - x₃ = 0
x₃ ∈ ℝ


DEFINITION: A SUBSET $W \subset \mathbb{R}^n$ IS A SUBSPACE OF $\mathbb{R}^n$, $W < \mathbb{R}^n$ IF
(a) $\underline{0} \in W$
(b) $\underline{x}, \underline{y} \in W \rightarrow \underline{x} + \underline{y} \in W$.
(c) $\underline{x} \in W$, $c \in \mathbb{R} \rightarrow c\underline{x} \in W$.


THEOREM: THE IMAGE AND KERNEL OF $\underline{x} \mapsto A\underline{x}$ ARE SUBSPACES.
DEFINITION: A LIST OF VECTORS $\underline{v}_1, \ldots, \underline{v}_k \in \mathbb{R}^n$ ARE LINEARLY INDEPENDENT IF ANY RELATION
$c_1\underline{v}_1 + \cdots + c_k\underline{v}_k = \underline{0}$, $c_1, \ldots, c_k \in \mathbb{R}$ REQUIRES $c_1 = \cdots = c_k = 0$.


WE SAY $\underline{v}_1, \ldots, \underline{v}_k$ ARE A BASIS FOR $\mathbb{R}^n$
IF • LINEARLY INDEPENDENT
• $\text{SPAN}\{\underline{v}_1, \ldots, \underline{v}_k\} = \mathbb{R}^n$.